

**FRACTIONAL STOCHASTIC ACTIVE SCALAR EQUATIONS
GENERALIZING THE MULTI-D-QUASI-GEOSTROPHIC &
2D-NAVIER-STOKES EQUATIONS.
-SHORT NOTE-**

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ABSTRACT. We prove the well posedness: global existence, uniqueness and regularity of the solutions, of a class of d -dimensional fractional stochastic active scalar equations. This class includes the stochastic, d D-quasi-geostrophic equation, $d \geq 1$, fractional Burgers equation on the circle, fractional nonlocal transport equation and the 2D-fractional vorticity Navier-Stokes equation. We consider the multiplicative noise with locally Lipschitz diffusion term in both, the free and no free divergence modes. The random noise is given by an Q -Wiener process with the covariance Q being either of finite or infinite trace. In particular, we prove the existence and uniqueness of a global mild solution for the free divergence mode in the subcritical regime ($\alpha > \alpha_0(d) \geq 1$), martingale solutions in the general regime ($\alpha \in (0, 2)$) and free divergence mode, and a local mild solution for the general mode and subcritical regime. Different kinds of regularity are also established for these solutions.

The method used here is also valid for other equations like fractional stochastic velocity Navier-Stokes equations (work is in progress). The full paper will be published in Arxiv after a sufficient progress for these equations.

Keywords: Scalar active equations, stochastic quasi-geostrophic equation, fractional stochastic vorticity Navier-Stokes equations, fractional stochastic Burgers equations, cylindrical Wiener process, Q -Wiener process, free divergence mode, martingale solution, mild solution, global and local solutions, Riesz transform, Hilbert transform, fractional operators, subcritical, critical and supercritical regimes, γ -radonifying operators, UMD Banach spaces of type 2.

Subclass[2000]: 58J65, 60H15, 35R11.

1. INTRODUCTION

The class of active scalar equations has been introduced as a set of simplistic models for many complex phenomena in fluid dynamics. Classical examples are the 2D Euler equation in vorticity form and Burgers equation. Non classical examples are equations with fractional dissipation, such as the quasi-geostrophic equation, the fractional dissipative nonlocal transport equation and the fractional Burgers equation. But unexpectedly, it immediately turned out that, questions related to this class are not trivial and far to be achieved. The fractional scalar equations showed a great mathematical and practical

potential in the understanding and in the modeling of many complex phenomena, see for short list, [3, 4, 15, 17, 24, 25].

In this work, we consider a class of d -dimensional active scalar equations driven by a fractional operators and perturbed by Gaussian random noises. The representative equation is formally given by,

$$(1.1) \quad \begin{cases} \theta_t(t, x) &= -\nu(-\Delta)^{\alpha/2}\theta(t, x) + u(t, x) \cdot \nabla\theta(t, x) + G(t, \theta)\xi(t, x), \\ u &= (u_j)_j, \quad \theta(0, \cdot) = \theta_0(\cdot), \end{cases}$$

where, $0 < \alpha \leq 2$, $\nu > 0$ are real parameters called the fractional order respectively the viscosity. The spatial coordinate $x \in \mathbb{T}^d$, with \mathbb{T}^d being the d -dimensional Torus, $d \geq 1$ and $t \geq 0$ is the time. The operator $(-\Delta)^{\frac{\alpha}{2}}$, denoted bellow by A_α , is the fractional power of minus Laplacian. The function $\theta_0(\cdot)$ is the initial condition and ξ is a random force. The variable θ represents the potential temperature and the vector field u represents the fluid velocity. The velocity u is determined by the function θ via the stream function ψ . This fact, contrarily to the passive dynamics, makes the dynamics active. In particular, for

$$(1.2) \quad u = \left(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\right) = \nabla^\perp\psi,$$

Equation (1.1) is called 2D fractional stochastic vorticity Navier-Stokes equation, respectively, stochastic surface quasi-geostrophic, if

$$(1.3) \quad \theta = \Delta\psi, \quad \text{respectively} \quad (-\Delta)^{\frac{1}{2}}\psi = \theta.$$

One possible generalization of the above example, is what called the modified stochastic quasi geostrophic equation. This latter is obtained by considering, see e.g. [5, 6, 10, 11],

$$(1.4) \quad \theta = (-\Delta)^{\frac{\gamma}{2}}\psi, \quad 1 \leq \gamma \leq 2.$$

In this work, we are dealing with Equation (1.1) with nonlinear term being a generalization of the relation (1.4), for $d \geq 1$, $\gamma \geq 1$ and u could be or not of divergence free. This equation covers the fractional stochastic Burgers equation on the circle, the d D stochastic quasi geostrophic equation, the 2D fractional stochastic vorticity Navier-Stokes equation, d -dimensional stochastic transport equation with nonlocal coefficients and equations in compressible fluids, such as, the vorticity Navier-Stokes equation with no-zero divergence.

The novelty in this work, concerns both the equation and the techniques. In fact, to the best knowledge of the author, this class of equations is introduced and studied here for the first time, in both deterministic and stochastic versions. The idea and the techniques used to prove the global existence of the mild solution for the subcritical regime and free divergence mode are also new. In particular, we establish a critical threshold, $1 \leq \alpha_0(d) < 2$, for which a unique global mild solution exists for this mode. Other related results are also obtained for larger threshold, $\alpha_0(d, q)$, with $q \geq 2$ is the integrability index. The martingale solution is obtained for the general regime ($\alpha \in (0, 2)$) and the free divergence mode by generalizing the Hilbert space techniques to the Banach space setting. The local mild solution is proved for

the general mode and the subcritical regime by construction. Different kinds of regularity are also established for these solutions.

Let us fix a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, W)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is a filtration satisfying the usual conditions, i.e. $(\mathcal{F}_t)_{t \geq 0}$ is an increasing right continuous filtration. The process $W := (W(t), t \in [0, T])$ is a mean zero Gaussian process defined $(\Omega, \mathbb{F}, \mathbb{P}, \mathcal{F})$, such that the covariance function is given by:

$$(1.5) \quad \mathbb{E}[W(t)W(s)] = (t \wedge s)Q, \quad \forall t, s \geq 0,$$

where Q is a nonnegative operator either of trace class on $L^2(\mathbb{T}^d)$ or $Q = I$. We rewrite equation (1.1) as an abstract evolution equation of type,

$$(1.6) \quad \begin{cases} d\theta(t) = (-\nu A_\alpha \theta(t) + B^{\sigma, \gamma}(\theta(t))) dt + G(t, \theta(t)) dW(t), & 0 < t \leq T, \\ \theta(0) = \theta_0, \end{cases}$$

where $\{\sigma, \gamma\}$ are two parameters characterizing the mode and the regularity of the nonlinear term. In particular, we are interested in three categories; C_a for the free divergence mode, C_b and C_c for the no free divergence mode with different regularities.

Let us, before announcing the assumptions on the diffusion term G and on the initial data θ_0 , fix a parameter δ as follow,

- (a) $\delta \geq 0$, if $(\sigma, \gamma) \in C_a \cup C_b$.
- (b) $\delta > \frac{1}{2}$, if $(\sigma, \gamma) \in C_c$.

Assumption (A): We assume that, for $q \geq 2$ and δ given by either (a) or (b), the operator $G : H^{\delta, q}(\mathbb{T}^d) \rightarrow \mathcal{L}(L^2, H^{\delta, q})$, is a locally Lipschitz continuous and of linear growth map in the following senses,

For all $R > 0$, there exists a constant $C_R > 0$, s.t.

$$(1.7) \quad \|(G(u) - G(v))Q^{\frac{1}{2}}\|_{R_\gamma(L^2, H^{\delta, q})} \leq C_R |u - v|_{H^{\delta, q}}, \quad \forall |u|_{H^{\delta, q}}, |v|_{H^{\delta, q}} \leq R,$$

where $R_\gamma(L^2, H^{\delta, q})$ denotes the set of γ -radonifying operators from $L^2(\mathbb{T}^d)$ to $H^{\delta, q}(\mathbb{T}^d)$. There exists a constant $c > 0$, s.t

$$(1.8) \quad \|G(u)Q^{\frac{1}{2}}\|_{R_\gamma(L^2, H^{\delta, q})} \leq c(1 + |u|_{H^{\delta, q}}), \quad \forall u \in H^{\delta, q}(\mathbb{T}^d).$$

Assumption (B): Assume that the initial condition θ_0 is an \mathcal{F}_0 -random variable satisfying

$$(1.9) \quad \theta_0 \in L^p(\Omega, \mathcal{F}_0, P; H^{\delta, q_0}(\mathbb{T}^d)),$$

with $2 \leq q_0 < \infty$, $p \geq 2$ and δ is given by either (a) or (b) above. In the case $\delta = 0$, we allow $q_0 = \infty$.

The full paper is organized as follow, after the introduction, we give in Section 2, the rigorous formulation of Problem (1.1). In particular, we define the nonlinear part of the drift term, the different notions of solutions and in the end of that section, we announce the main results. In Section 3, we prove several lemmas to estimate the nonlinear term. Sections 4-6 are devoted to prove the main theorems of this work; Theorems 3.1-3.3.

2. DEFINITIONS.

2.1. Definitions of solutions. In this subsection, we give some definitions of solutions for stochastic partial differential equations. We are interested in solutions as processes in UMD Banach spaces of type 2. In particular in $H^{\delta,q}(\mathbb{T}^d)$, $2 \leq q < \infty$, $\delta \geq 0$. In the cases where such generality is not needed, as in Definitions 2.3 and 2.4 bellow, we restrict ourselves only on the spaces considered in this work.

Definition 2.1. *Let $p \geq 2$ and X be an UMD-Banach space of type 2. Assume that $\theta_0 \in L^p(\Omega, \mathcal{F}_0, P; X)$. Then an X -valued adapted stochastic process $(\theta_t, t \in [0, T])$, is called mild solution of Equation (1.6), iff*

- *there exists a Banach space, $X_1 \hookleftarrow X$, such that*

$$(2.1) \quad \theta(\cdot, \omega) \in C([0, T]; X_1) \cap L^\infty(0, T; X),$$

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$$(2.2) \quad \mathbb{E} \sup_{[0, T]} (|\theta(t)|_X^p) < \infty$$

- *and $\forall t \in [0, T]$,*

$$(2.3) \quad \theta(t) = e^{-A_\alpha t} \theta_0 + \int_0^t e^{-A_\alpha(t-s)} B(\theta(s)) ds + \int_0^t e^{-A_\alpha(t-s)} G(\theta(s)) W(ds). \text{ a.s.}$$

Definition 2.2. *Let X be an UMD-Banach space of type 2. Assume that $\theta_0 \in L^p(\Omega, \mathcal{F}_0, P; X)$. A local mild solution of Equation (1.6) is a couple (θ, τ_∞) , where $\tau_\infty \leq T$ a.s. is a stopping time and $(\theta(t), t \in [0, \tau_\infty))$ is an X -valued adapted stochastic process such that*

- *there exists a Banach space, $X_1 \hookleftarrow X$, such that*

$$(2.4) \quad \theta(\cdot, \omega) \in C([0, \tau_\infty); X_1),$$

- *there exists an increasing sequence of stopping time $(\tau_n)_{n \in \mathbb{N}}$, s.t. $\tau_n \nearrow \tau_\infty$.*
- *For all $n \in \mathbb{N}$, the process $(\theta(t \wedge \tau_n), t \in [0, T])$ satisfies the stopped (2.3) equation, i.e. for all $n \in \mathbb{N}$ and $\forall t \in [0, T]$, the following equation is satisfied*

$$(2.5) \quad \begin{aligned} \theta(t \wedge \tau_n) = e^{-A_\alpha(t \wedge \tau_n)} \theta_0 &+ \int_0^{(t \wedge \tau_n)} e^{-A_\alpha(t \wedge \tau_n - s)} B(\theta(s \wedge \tau_n)) ds \\ &+ \int_0^t e^{-A_\alpha(t-s)} 1_{[0, \tau_n)}(s) G(\theta(s \wedge \tau_n)) W(ds). \text{ a.s.} \end{aligned}$$

Definition 2.3. *Let $2 \leq q < \infty$. Assume that $\theta_0 \in L(\Omega, \mathcal{F}_0, P; L^q(\mathbb{T}^d))$. An L^q -valued adapted stochastic process $(\theta_t, t \in [0, T])$, is called weak solution of Equation (1.6), iff*

$$(2.6) \quad \theta(\cdot, \omega) \in L^\infty(0, T; L^q(\mathbb{T}^d)) \cap L^2(0, T; H^{\frac{\alpha}{2}, 2}(\mathbb{T}^d)) \cap C([0, T]; H^{-\delta', q}(\mathbb{T}^d)) \text{ a.s.}$$

where $\delta' \geq 1 \max\{\alpha, 1 + \frac{d}{q^*}\}$ with $q^* = \frac{q}{q-1}$ and such that for all $\varphi \in D(A_{q^*}^{\frac{\eta}{2}})$ with $\eta \geq 2 \max\{1 + \frac{d}{q}, \frac{\alpha}{2} - \frac{d}{2} + \frac{d}{q^*}\}$, we have

$$(2.7) \quad \begin{aligned} \langle \theta(t), \varphi \rangle &= \langle \theta_0, \varphi \rangle + \int_0^t \langle \theta(s), A_{q^*}^{\frac{\alpha}{2}} \varphi \rangle ds + \int_0^t \langle \mathcal{R}^{\gamma, \sigma} \theta(s) \cdot \nabla \varphi, \theta(s) \rangle ds \\ &+ \left\langle \int_0^t G(\theta(s)) dW(s), \varphi \right\rangle. \text{ a.s.} \end{aligned}$$

Definition 2.4. The multiple $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, \mathbb{F}^*, W^*, \theta^*)$, where $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, \mathbb{F}^*, W^*)$ is a stochastic basis with W^* being a Q -Wiener process of trace class and $\theta^* := (\theta^*(t), t \in [0, T])$ is an adapted stochastic process, is called a martingale solution of Equation (1.6), iff θ^* is a solution of Equation (1.6) in the sense of Definition 2.3 on the basis $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, \mathbb{F}^*, W^*)$.

3. RESULTS.

The main results of this work are

Theorem 3.1. [Free divergence mode & subcritical regime] Let $d \in \{1, 2, 3\}$ and $T > 0$ be fixed and let $\alpha \in (\alpha_0, 2]$, with

$$(3.1) \quad \alpha_0 = \alpha_0(d) := 1 + \frac{d-1}{3}.$$

Assume $(\sigma, \gamma) \in C_a$, G and θ_0 satisfying Assumptions (\mathcal{A}) respectively (\mathcal{B}) , with $\max\{2, \frac{d}{\alpha-1}\} \leq 1$ $q_0 \leq \infty$. Then Equation (1.6) has a unique global mild solution, $(\theta(t), t \in [0, T])$, in the sense of Definition 2.1, with $X = L^q(\mathbb{T}^d)$, $p = q$, $X_1 = H^{-\delta', q}(\mathbb{T}^d)$, where δ' is given in Definition 2.3 and

- (1) for $d = 1$, then $\max\{2, \frac{1}{\alpha-1}\} \leq 1$ $q \leq \infty$ q_0 ,
- (2) for $d \in \{2, 3\}$, then $\frac{d}{\alpha-1} < q \leq \frac{3d}{d-1} \leq \infty$ q_0 .

The solution also satisfies,

$$(3.2) \quad \mathbb{E} \left(\sup_{[0, T]} |\theta(t)|_{L^q}^q + \int_0^T |\theta(t)|_{H^{\beta, q}}^2 dt \right) < \infty,$$

where $\beta \leq \frac{\alpha}{2} - \frac{d}{2} + \frac{d}{q}$. If in addition, Assumption (\mathcal{B}) is satisfied for $\delta > 0$, then

$$(3.3) \quad \theta(\cdot, \omega) \in L^\infty(0, T; H^{\delta_1, q}(\mathbb{T}^d)),$$

with $0 \leq \delta_1 \leq \min\{\delta, \alpha - 1 - \frac{d}{q}\}$ (q satisfies either (1) or (2)).

Furthermore, for $d \in \{2, 3\}$, the solution exists in the following cases;

case 1. $\frac{3d}{d-1} \leq q \leq \infty \min\{q_0, \frac{2d}{d-\alpha}\}$ and $d - 2\frac{d}{q} < \alpha \leq 2$.

case 2. $\frac{d}{\alpha-1} < q \leq q_0 \leq \frac{3d}{d-1}$ and $1 + \frac{d}{q} < \alpha \leq 2$.

Theorem 3.2. *[General mode \mathcal{E} subcritical regime] Let $d \in \mathbb{N}_0$, $T > 0$ and $p \geq 2$ be fixed. Assume $(\sigma, \gamma) \in C_a \cup C_b \cup C_c$, G and θ_0 satisfying Assumptions (\mathcal{A}) respectively (\mathcal{B}) with $\max\{2, \frac{d}{\alpha-1}\} \leq_1 q_0 \leq \infty$ and that $\alpha \in (1 + \frac{d}{q}, 2]$ with $\max\{2, \frac{d}{\alpha-1}\} \leq_1 q \leq \infty$ q_0 . Then Equation (1.6) has a local mild solution in the sense of Definition 2.2, satisfying*

$$(3.4) \quad \theta \in L^p(\Omega; L^\infty(0, \tau_\infty; H^{\delta_1, q}(\mathbb{T}^d))) \cap L(\Omega; C([0, \tau_\infty); H^{-\delta'', q}(\mathbb{T}^d))),$$

with δ_1 given in Theorem 3.1 and $\delta'' \geq \alpha + 1 + \frac{d}{q} - \delta$.

Theorem 3.3. *[General regime \mathcal{E} free divergence mode] Let $d \in \mathbb{N}_0$, $\alpha \in (0, 2]$ and $T > 0$ be fixed. Assume $(\sigma, \gamma) \in C_a$, G and θ_0 satisfying respectively (1.8) and (\mathcal{B}) . Then Equation (1.6) has a martingale solution, $(\theta(t), t \in [0, T])$, in the sense of Definition 2.4, satisfying (2.6), (2.7) and (3.2), with $q = 2$.*

Moreover, the solution $(\theta(t), t \in [0, T])$ satisfies (2.6), (2.7) and (3.2), for $d(1 - \frac{2}{q}) \leq \alpha \leq 2$ and either

- $d \leq \alpha$ and $2 \leq q \leq \infty$ q_0 , or
- $d > \alpha$ and $2 \leq q \leq \infty$ $\min\{q_0, \frac{2d}{d-\alpha}\}$.

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